**Partial Differential Equations**

**Consider a simple 1D PDE**

This PDE can be solved when you realize what it means, exactly. Each point on the profile moves with velocity v starting at 0 . So the profile piece will have moved to a time later. .  
which basically just says that we shift the curve over to the right vy an amount vt. Consider a more complicated one,

This one can be similarly solved. In this case though, the velocity is time dependent so that the wave gets distorted as it moves with a different velocity at each time. Still, the total distance the point travels is integral(vdt, .

Consider a more complicated one,

This one can be similarly solved. In this case though, the velocity is time dependent so that the wave gets distorted as it moves with a different velocity at each time. We can picture this below. And we can see that the u at x and time t is just at the point  
x 0 that evolved into x at by time t

![](data:application/octet-stream;base64,)

So this consideration enables us to postulate that , where is the point which will evolve into by time . So let's try one.

Well the velocity of the points is:

where x 0 is the initial position. Requiring that we have:

So the solution should be:

Checking ...

So it checks out. OK, now let's look at a worse one.

The inhomogeneity has units of inverse time. So it is some sort of scattering rate. But how to understand its effect?

Consider writing it as:

So the rate of accumulation is given by the convection term + the rate of accumulation . So I would say that the amount of at and is equal to the sum of two terms - that coming from the convection, which we can calculate, and also that coming from the external source. As we trace the path of the particle backwards through time to , it seems that the particle will acquire more coming from the source at a rate dependent on its location and time. So the net that will have accumulated at point at time will be the amount that has been carried into that point by the particle traveling through the path.  
 such that

Lets try,

We already have that

So we would expect our solution to be

Check it out...

Now let's look at general first order equations

Lets think about what is going on to see if can use same paradigm.

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So we see that x 0 will will develop along some path as before. And we'll assume to be later determined. So what is at time t 1 ? Well at x 0 it is . Then at time t 1 , etc., it will be:

So generally we can see that:

So we can solve the related PDE which is:

and the coordinate is related to the velocity of the point, which is given by:

So we must solve the linear ODE in terms of X and t . And then find what X is in terms of x and t by solving the trajectory equation. Now let's look at general first order equations from a different perspective which will yield the same results.

So we will change variables to , and . So that:

This requires that:

This enables us to get the dependence of and on , but not . The other condition we'll employ is that we want when , then , and when , then . This is so the initial conditions transfer nicely. So we have:

Consider the PDE:

So we have:

And this could be determined somehow in principle. So then our differential equation comes to:

is our general solution. To impose the initial condition we need to do:

So our solution is:

which is precisely what we obtained before. OK now let's try a non-linear equation. Such equations appear in the Schrodinger equation and also in the study of shock waves:

Lets think about what is going on to see if can use same paradigm.

![](data:application/octet-stream;base64,)

So we see that x 0 will will develop along some path as before. And we'll assume to be later determined. So what is at time t 1 ? Well at x 0 it is . Then at time t 1 , etc., it will be:

where dt is the interval of time between , etc., which is assumed to be constant. Now note that is just equal to , and hence it appears that since the difference in in each successive time interval is just , then dv is also just . So we have:  
which can be put in terms of itself. Also we have that and . So altogether we can find the path of in this phase space:

So we would solve for in terms , and . Then I think we would have to solve for in terms of using the fact that . Then we would have completely defined in terms of and principle. And then our solution could be written.

Actually this simplifies a bit more, since:

For example, consider:

Then the path is governed by:

Now so:

Supposing that , like in the PDE book, then this equation reduces to:

Therefore the solution is:  
Consider another:

Then the path is governed by:  
Now X(t) = x so:

Therefore the solution is:

OK let's test it out...

So it works! Finally, one more non-linear equation

This could be derived from the previous one. For instance, taking a derivative to x gives

And then defining u\_x as (1/2) v\_x:

which is of the same form as the problem above. Once we solve for , then we'd have to integrate to get .

But this doesn't completely determine . So we plug it back into the original PDE.

To solve for . Note this equation must hold for all x and so can pick any that's convenient.  
Lets think about what is going on to see if can use same paradigm.

![](data:application/octet-stream;base64,)

So we see that x 0 will will develop along some path as before. And we'll assume to be later determined. So what is at time ? Well at it is . Then at time , etc., it will be:

What is ? It is the . I suppose we could relate this to since . But this may not be rigorous and in any event, I'm missing the Lagrangian. Let's try to look at it from the perspective of a stationary x . Let's take f out of the picture for now. Let's do:

**More generally**

Suppose we have the equation:

We may interpret this as an equation in the following manner. Suppose that x and y were functions of some variable s , and therefore that u was implicitly as well so that . Then du/ds would look like,  
Comparing with the top equation, we'd have the three equations:

And we could write the initial conditions as:

So then this would define a solution of in terms of and , which would need to be inverted in terms of and .

**Or another explanation**

Consider the PDE

The general solution to this equation is:  
 const.  
What is ? Now we'll write this PDE in terms of itself. So we have:  
and so we have, holding constant, and then constant respectively:  
So then plugging this into our PDE we have, for f :  
Now we'll just show that is perpendicular to the solution surface. Consider two neighboring points and which lie on the hyper-surface defined by the above equation. Then these two points are 'solutions' of the PDE, and in any event, to first order we have:

So the gradient vector:

is perpendicular to the surface. But then rewriting the differential equation for we have:  
So then we can say that the former vector lies in the surface of the solution. So pick a point in the solution surface, say . Then if  
then the differential vector will lie in the solution surface. So then we have the equation  
which follows from the other characteristic equations anyway. And this equation traces out curves which lie in the solution surface. These are called characteristic curves. These characteristic curves have two degrees of freedom. An integral of the characteristic curve is a function which is constant along the curve, just like is.

**Second order PDE's**

**I. Heat Equation and derivation from continuity equation**

Continuity equation: where u is the density of stuff and is the flux density, a vector field comparable to J, the current density, which associatates each point in space with a vector pointing in the direction of flow whose magnitude

For example, consider: is the differential amount of stuff passing through a differential area, divided by that area, per unit of time. is the source density

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Some examples of 's are |  | velocity | example of |  | radioactive decay |
|  |  | convection |  |  |  |

For heat equation: energy\_density and then divide by volume to get

In one dimension

**BC in 1-D:**

Case 1 : area of interest is bounded by two regions whose temperatures, g 0 and g 1 , are specified  
equate the two conditions and arrive at

Case 2: temperature is specified at the ends

Case 3: Flux is specified at the ends

Only when the BC are homogenous will you get a sturm-louiville problem after separation of variables so it is usefull to make a substitution  
 and solve for the A , and B . This substitution will possibly add an inhomogeneity to the PDE and will definitely change the initial conditions  
for case and

**Explorations of the Diffusion equation**

Is the diffusion equation when we have a position dependent diffusion constant, as well as a position dependent convenction term. The position dependent convention term would simply mean that the diffusive medium as a position dependent velocity. The position dependent diffusion constant would mean that the medium composition is position dependent.

When do a separation of variables approach - you get two ODE`s to solve. The eigenvalue equation will produce some eigenvalues, , and the general solution will follow. Note that for long times, only the GS solution will remain.

Another way to work with the convenction term is to write the solution as a product of two terms - one dealing with diffusion and the other with convection. For simplicity we'll suppose that and are constants.  
Now want to choose c in such a way that the equation in d satifies diffusion.  
Both bracket terms must disappear. The first bracket term will solve for the x dependence of c , and the second will solve for the t dependence.

Note the interesting fact that the convection part seemingly travels at half the speed its supposed to. The total solution however will be a function of . Also, this would still have been quasi-easy to do if the coefficients were position dependent. Let's see what the solution of the equation with only the convection part gives us.  
where l've rescaled the eigenvalue. Note that this is the wrong form. The diffusion part (which first of all contains unlike this one, is travelling at half the speed this one is. But this equation is instructive in that it suggests (yes it does) that we should change variables to, say:  
Then we'd get a diffusion equation in , I believe. Now let's look at the time independent solution.  
We see that we get only half the required x part. What would we get if we factor u into a product of this term and another?  
We seem only to change the direction of the convection. This seems to indicate that if we substitute in the sqrt of the term we might eliminate convection completely. So let's make the following partition instead.  
So substitution of this term turns it into a lateral diffusion problem, which can be solved by making the usual factorization. Note that this is simply an imaginary time Schrodinger equation.  
Now let's consider making the same substitution, but in an equation with an -dependent velocity field.

This obviously isn't nicer at all, presumably is because isn't constant; otherwise the equation would reduce to the diffusion equation. OK, since the Schrodinger equation is equidimensional in , let's make the substitution , to get an autonomous equation.

Can't exactly make the desired autonomous in u substitution. But try separation of variables and then do it.

Alright now lets put in the substitution for the X part.

Now make the substitution.   
This equation is solvable with GF techniques. I'll absorb the in and call it .  
Now suppose we do the conventional separation of variables. We'll get a solution of the following form.  
where the coefficients are chosen to satisfy the initial condition. Of course, the eigenfunctions must satisfy the boundary conditions. Now if we want the long time behavior, we'd simply write,  
What if we have a continuum of states?  
Well, I think l'd need to know the form for the first few eigenfunctions, and eigenvalues. I can approximate the situation for large t (small k ) then. I don't need all eigenfunctions to get the coefficients, just the relevant small k ones.

**Schrodinger Equation with delta function IC**

Now suppose that we have delta function IC,

The solution of the equation, neglecting the part, and with the desired IC is of the form:  
Let's determine what beta is:

Now equating these two sides, we'd have  
OK so that's reassuring. Our solution is:

Now what happens if we factor out this term from the equation.

So we get the equation above. Perhaps we could say we're interested only in long times and solve the equation below, along with the IC.

Going to consider a bunch of PDE’s that show up in Physics.

**Surface PDE**

Consider a PDE like this:



Now consider a solution u(x,y), which we can consider to define a surface in x, y space. Recall that if we have a surface defined by g(x,y,z) = constant, then the normal vector to this surface is ∇g. We know this because,



And so for any displacement along the surface, i.e., tangent to the surface, d**r**, we have ∇g is perpendicular to it. So the normal vector to this surface u(x,y) is ∇[u(x,y) – u],



Now observe that we can write the PDE as:



So this means the vector tangent to the solution surface is:



Now consider a parameterized curve [x(t),y(t),u(t)]. A vector tangent to the curve would be α[dx/dt, dy/dt, du/dt], where α could be anything. So it follows that this must be equal to the tangent curve. And so we can say, where the constant α has been absorbed into the parameter t.



Let’s do an example. Let’s take the surface u = x2 + y2. Then,



And so should have:



Solution is:



ugly. Could say,



So then,



Looks like the latter is a homogenous solution. But there doesn’t seem to be a general way to figure out what it should be.

**Convection PDE**

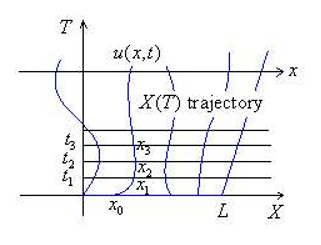
Now let’s take a general look at convection/source PDE’s. Diffusion would be neglected, as this requires a double derivative term.

**Linear**

OK so consider a general linear PDE:



It looks like we can consider the solution as having stream lines that flow from our initial condition (being depicted in just 1D), while picking up ‘debris’ as it goes along.



So we interpret the p and q terms as the velocities of the stream at any point and time. Continuing along that point of view, we can write the equation as:



This compares favorably to the current equation ∂n/∂t + ∇(nv) = 0. In this context, we can see that we could interpret r(x,y,t) as something like ∇∙v. And f(x,y,t) would just be some source term. But we don’t need to push the analogy. The important thing is that we can interpret the terms. At point (x,y) and time t, the concentration changes because of the stream velocity vx = p and vy = q. If there is a concentration gradient there, then in the x-direction, a different amount of u will be carried in as is carried out and so u will change. Same thing for the y-direction. The r(x,y,t)u term represents an amount of change proportional to u. Maybe radioactive decay or something (which would happen proportional to u, or, negative u). And the f(x,y,t) is a source term – the rate at which more u is being dumped in (or taken out) externally. We can solve this equation by thinking of it slightly differently, in terms of stream lines. Consider the a point initially at x0, y0 (t = 0). If we follow this point in the stream, with a trajectory x(t), y(t), which we can determine from the velocity components, vx = p and vy = q:



Along this stream the concentration would be constant, except for the ru and f terms. So along this stream, we’d actually have the concentration following the ODE:



Solving these equations will give us x(x0,y0,t), y(y0,t), and u(t). What do we do with them? What is the concentration at (x,y) at time t? Well, this is u(x0,y0,t) for the particular x0,y0 which reached the point x,y at time t. If we invert the x(x0,y0,t) and y(x0,y-,t) equations to get x0(x,y,t) and y0(x,y,t), then we can get u(x,y,t) via u(x0(x,y,t),y0(x,y,t),t).

**Example**

Say we have radioactive material in the atmosphere following a nuclear explosion. Its concentration along the x direction is initially given by:



The wind is blowing in the x direction with speed v = 15 m/s. And the radioactive material decays with decay constant α = 0.0025s-1. So an equation which describes the concentration, as a function of time, presuming convection is the predominant transport mechanism, is:



So what will be u(x,t)? And say we’re at a x = 10 000 m down the road. Will the radiation concentration there ever be greater than the critical amount umax = 100m-1? So first, we need to solve the trajectory equations.



And then we say,



Then inverting the x(x0,t) equation, we have:



And so,



Just to verify, the intial conditions are satisfied, and filling into the PDE, we have:



So it works. Now, say we’re interested in the point x = x1. And we want to know how much radiation exposure it is getting. Maybe we want to know the maximum value of the radiation its getting. So we’d maximize u(x1,t) w/r to time.



And the concentration at that time is:



Let’s plug in the numbers. First we find,



and then,



So the radiation concentration doesn’t exceed 100.

**Simple Nonlinear**

Let’s take a look at this guy:



We may proceed as before, and write:



But problem, obviously, is that we don’t know what u is apriori. So we can differentiate these equations, and use what we know du/dt to be. And we have to supplement with knowledge of the initial values of dx/dt, and dy/dt, which follow straight from those equations themselves. So we have:



and,



Then the general solution is constructed as before.

**General Nonlinear**

Let’s consider more generally,



We can presume to still say:



All total, these constitute a self-consistent set of equations. Yeah this should work. And general solution would work just like before.

**Diffusion PDE**

Just occurred to me that convection/source equations can be thought of arising from points following these deterministic trajectories. Diffusion equations can be thought of as arising from points following stochastic trajectories – i guess in particular, ones governed by Brownian variables? Can all PDE’s be reduced to some sort of corresponding trajectory problem?

**G. Perturbation Expansions**

For ODE's, and PDE's

Say you have . and also some BC, like, say, 

Then you will, as before, expand your solution in powers of , and solve to each order both the ODE, and the BC.

If you add a perturbation term like 

Not only can you solve perturbed differential equations, but you can also solve perturbed BC.  
Suppose you have BC like:

![](data:application/octet-stream;base64,)



You can expand the perturbed BC in powers of .  


Then expand  in powers of , and solve the BC and PDE at each order.